Math 564: Real analysis and measure theory
Lecture 2

Sigma-algebras.

Def. Let X be a set. A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ of subsets of X is called an algebra (resp. σ-algebra) if $\emptyset \in \mathcal{A}$ and $\mathcal{A}$ is closed under complements and finite unions (resp. ctbl unions), hence also under finite (resp. ctbl) intersections.
A set X equipped with a σ-algebra $\mathcal{S}$ is called a measurable space, denoted $(X, \mathcal{S})$.

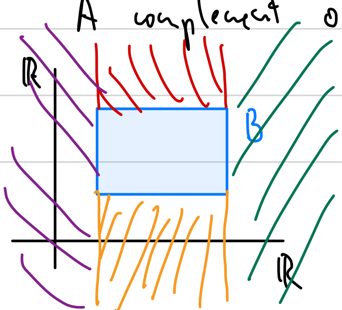
Examples.

(a) Let X be a set. The collection $\mathcal{A}$ of finite and co-finite sets is an algebra (because finite unions of finite sets are finite). The collection $\mathcal{S}$ of ctbl and co-ctbl sets is a σ-algebra. Also, the powerset $\mathcal{P}(X)$ is a σ-algebra.

(b) In a metric/topological space, the collection of clopen sets is a algebra, we call it the algebra of clopen sets.

(c) For a finite nonempty set A, the clopen sets in $A^{\mathbb{N}}$ are exactly the finite disjoint unions of cylinders, where the finiteness comes from the compactness of $A^{\mathbb{N}}$, HW.

(d) A box in $\mathbb{R}^d$ is a set of the form $B := I_1 \times I_2 \times \dots \times I_d$, where each I_j is a (potentially unbdd) interval, e.g. $(-7, \pi)$, $(0, 1]$, $[7, \infty)$, $(-\infty, \infty)$.
A complement of a box B is a finite disjoint union of boxes, so the collection of finite disjoint unions of boxes is an algebra.
(Also note that a finite union of boxes is a finite disjoint union of boxes.)



Observation. An arbitrary intersection of (σ-) algebras is a (σ-) algebra, i.e. if $\mathcal{A}_i$, $i \in I$, are (σ-) algebras then $\bigcap_{i \in I} \mathcal{A}_i$ is a (σ-) algebra.

This allows us to define:

Def. Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$. The ($\sigma$-) algebra generated by $\mathcal{C}$ is the smallest (σ-algebra) containing $\mathcal{C}$ ($\mathcal{P}(X)$ is a σ-algebra containing $\mathcal{C}$), namely:

algebra $\langle \mathcal{C} \rangle := \bigcap \{ \mathcal{A} : \mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra and $\mathcal{A} \supseteq \mathcal{C} \}$;

σ-algebra $\langle \mathcal{C} \rangle_\sigma := \bigcap \{ \mathcal{S} : \mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{S} \supseteq \mathcal{C} \}$.

Def. For a metric/topological space X, the σ-algebra $\mathcal{B}(X)$ generated by open sets is called the Borel σ-algebra and the sets in it are called Borel sets.

The definition of $\langle \mathcal{C} \rangle$ and $\langle \mathcal{C} \rangle_\sigma$ are top-down, and we now give their bottom-up equivalents:

Prop. Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$. Then

(a) $\langle \mathcal{C} \rangle = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$, where $\mathcal{C}_0 := \mathcal{C}$ and $\mathcal{C}_{n+1} := \{ B^c : B \in \mathcal{C}_n \} \cup \{ \bigcup_{i<k} B_i : B_i \in \mathcal{C}_n$ and $k \in \mathbb{N} \}$.

(b) $\langle \mathcal{C} \rangle_\sigma = \bigcup_{\alpha \in \omega_1} \mathcal{C}_\alpha$, where $\mathcal{C}_0 := \mathcal{C}$ and $\mathcal{C}_\alpha := \{ B^c : B \in \mathcal{C}_\beta, \beta < \alpha \} \cup \{ \bigcup_{i \in \mathbb{N}} B_i : B_i \in \bigcup_{\beta < \alpha} \mathcal{C}_\beta \}$.

ω_1 ← the smallest unctbl cardinal

Proof. (a) is HW, (b) is optional. $\square$

Observation. In a metric/topological space, for any ctbl basis $\mathcal{U}$, the σ-algebra generated by $\mathcal{U}$ is $\mathcal{B}(X)$ — all Borel sets.

Proof. Every open set $\mathcal{O}$ is a union of sets in $\mathcal{U}$, hence a ctbl union of sets in $\mathcal{U}$, hence $\mathcal{O} \in \langle \mathcal{U} \rangle_\sigma$, so $\langle \mathcal{U}_\sigma \rangle$ is a σ-algebra containing all open sets, hence $\mathcal{B}(X) \subseteq \langle \mathcal{U} \rangle_\sigma$. But also $\langle \mathcal{U} \rangle_\sigma \subseteq \mathcal{B}(X)$ because $\mathcal{U}$ is a collection of open sets. $\square$

Measures.

Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$. A function $\mu : \mathcal{C} \to [0, \infty]$ is said to be **finitely additive** (resp. **ctbly additive**) if

$$\mu\Big(\bigsqcup_{i<k} A_i\Big) = \sum_{i<k} \mu(A_i) \quad \text{whenever } k \in \mathbb{N},\ A_i \in \mathcal{C}, \text{ and } \bigsqcup_{i<k} A_i \in \mathcal{C}.$$

$$\Big(\text{resp. } \mu\Big(\bigsqcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mu(A_i) \text{ whenever } A_i \in \mathcal{C} \text{ and } \bigsqcup_{i\in\mathbb{N}} A_i \in \mathcal{C}.\Big)$$

Def. For a measurable space $(X, \mathcal{S})$, a **measure** on $(X, \mathcal{S})$ is a ctbly additive function $\mu : \mathcal{S} \to [0, \infty]$ with $\mu(\emptyset) = 0$. The triple $(X, \mathcal{S}, \mu)$ is called a **measure space**.

Caution. People also deal with **finitely additive measures** on algebras, but a finitely additive measure even on a σ-algebra is, in general, not a measure because it may not be ctbly additive.

A measure μ on a measurable space $(X, \mathcal{S})$ is called

- a **probability measure** if $\mu(X) = 1$.
- **finite** if $\mu(X) < \infty$.
- **σ-finite** if $X = \bigcup_{n\in\mathbb{N}} B_n$, $B_n \in \mathcal{S}$ and $\mu(B_n) < \infty$.